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Circle Properties

Circle Perimeter

Proof

By definition, the perimeter is the length of the circumference of the circle.

Let C be the perimeter.

Then:

$$\pi = \frac{C}{2r} \qquad \text{Definition of } \pi \text{ (pi)}$$

$$\leadsto \quad C = 2\pi r$$

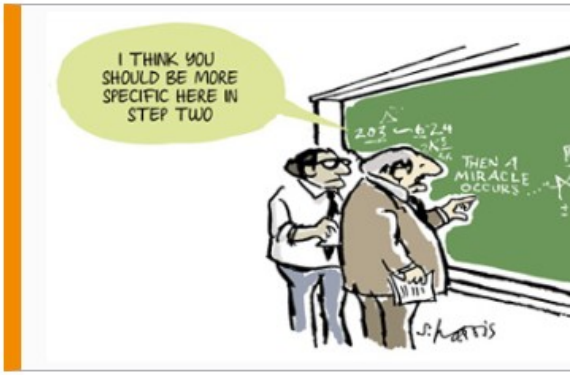
■

Circle Area

Proof 2

Proof by shell integration:

The circle can be divided into a set of infinitesimally



$$A = \int_0^r 2\pi t \, \mathrm{d}t \qquad \text{Perimeter of Circle}$$

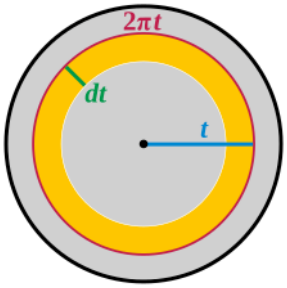
$$= \left[\pi t^2\right]_0^r$$

$$= \pi r^2$$

■

Onion proof [edit]

Using calculus, we can sum the area incrementally, partitioning the disk into thin concentric rings like the layers of an onion. This is the method of shell integration in two dimensions. For an infinitesimally thin ring of the "onion" of radius t, the accumulated area is $2\pi t\,dt$, the circumferential length of the ring times its infinitesimal width (one can approximate this ring by a rectangle with width=$2\pi t$ and height=dt). This gives an elementary integral for a disk of radius r.



Area of the disk via ring integration

$$\begin{aligned}\operatorname{Area}(r) &= \int_0^r 2\pi t\,dt \\ &= 2\pi\left[\frac{t^2}{2}\right]_0^r \\ &= \pi r^2.\end{aligned}$$

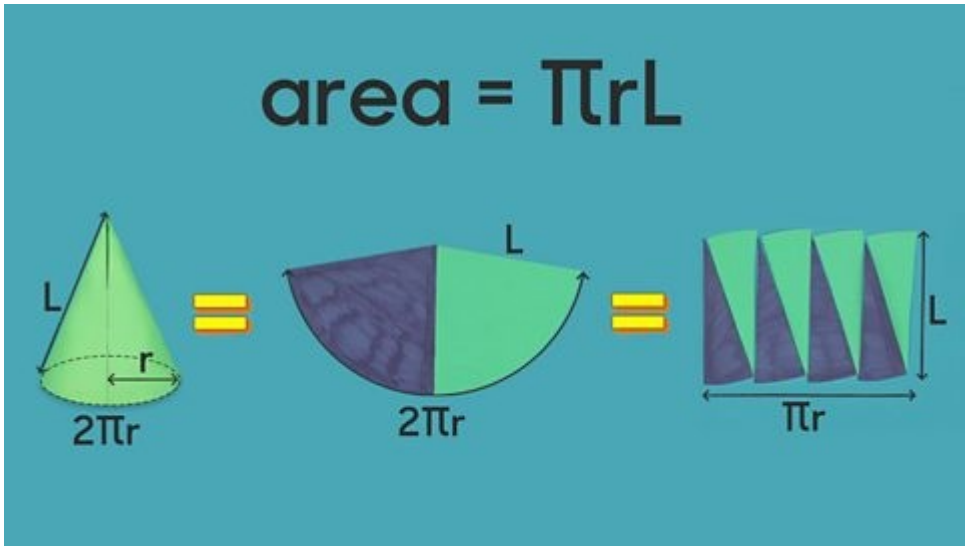
It is rigorously justified by the multivariate substitution rule in polar coordinates. Namely, the area is given by a double integral of the constant function 1 over the disk itself. If D denotes the disk, then the double integral can be computed in polar coordinates as follows:

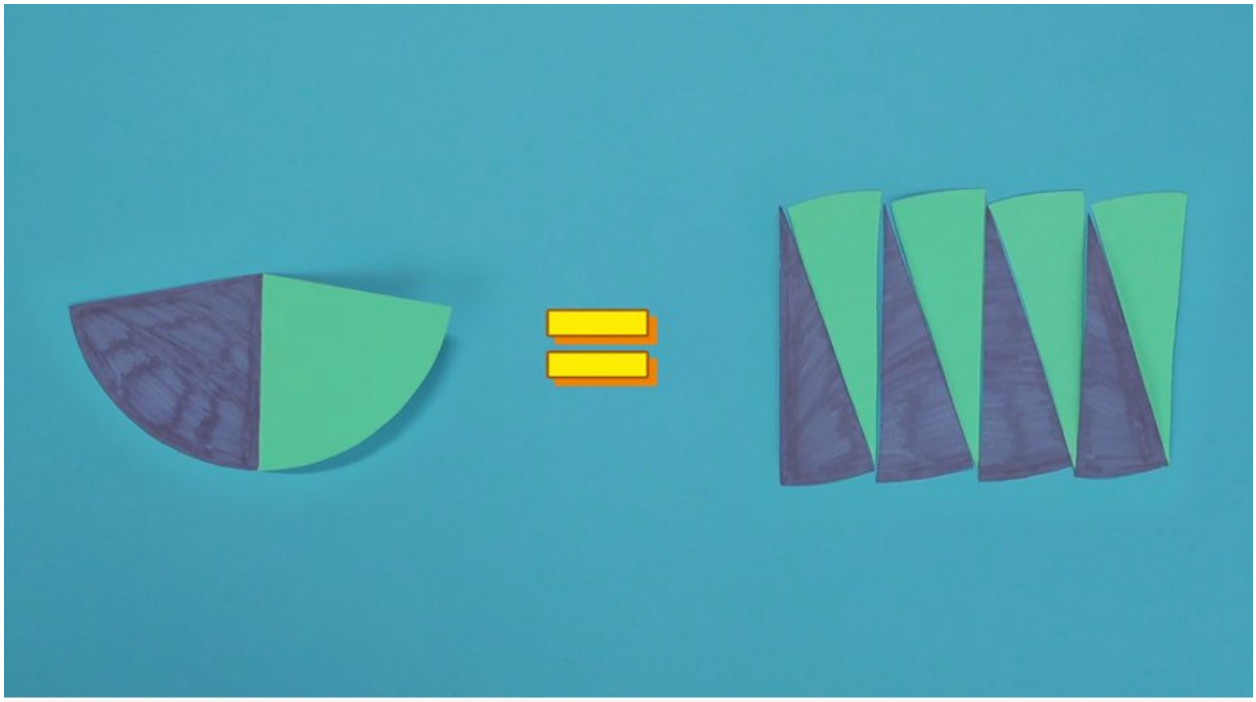
$$\begin{aligned}\operatorname{Area}(r) &= \iint_D 1\,d(x,y) \\ &= \iint_D t\,dt\,d\theta \\ &= \int_0^r \int_0^{2\pi} t\,d\theta\,dt \\ &= \int_0^r [t\theta]_0^{2\pi}\,dt \\ &= \int_0^r 2\pi t\,dt\end{aligned}$$

which is the same result as obtained above.

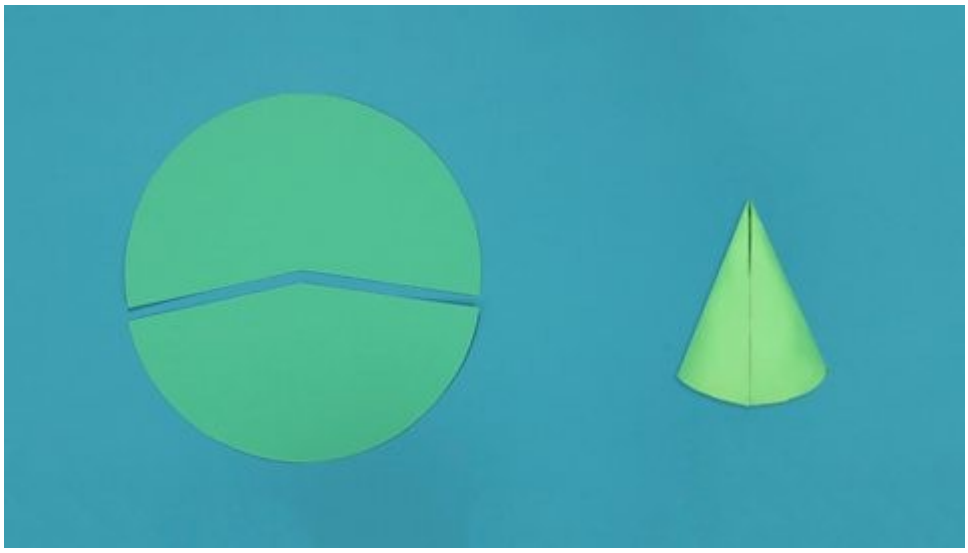
Cone Properties

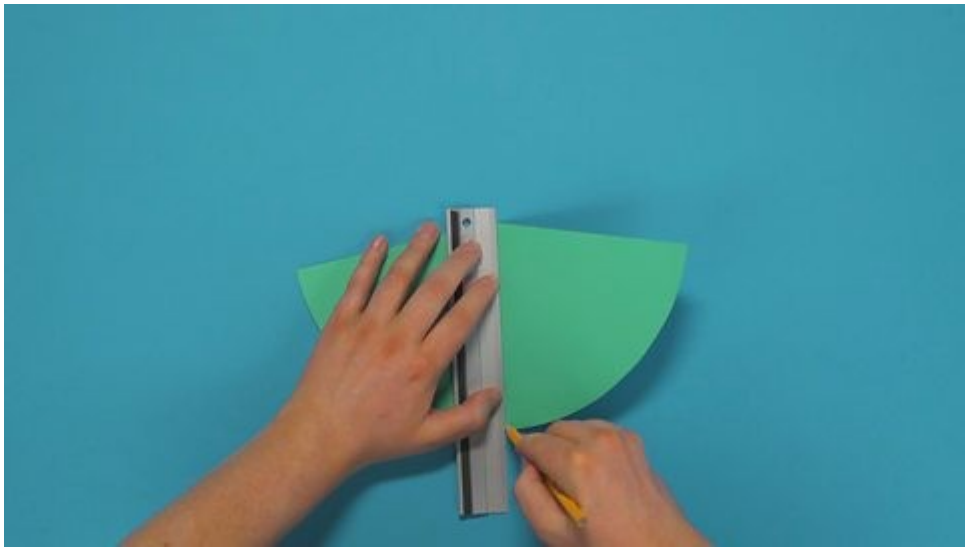
Cone Surface Area

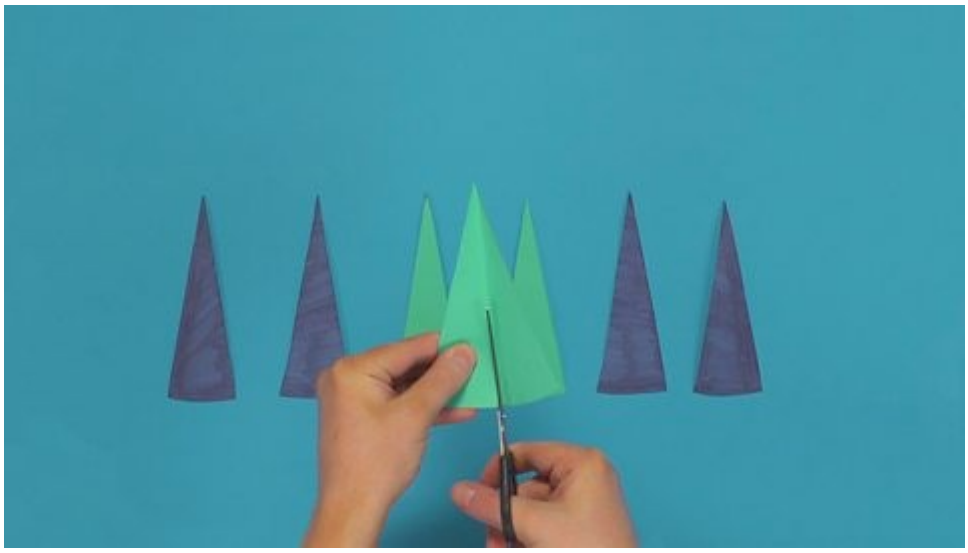


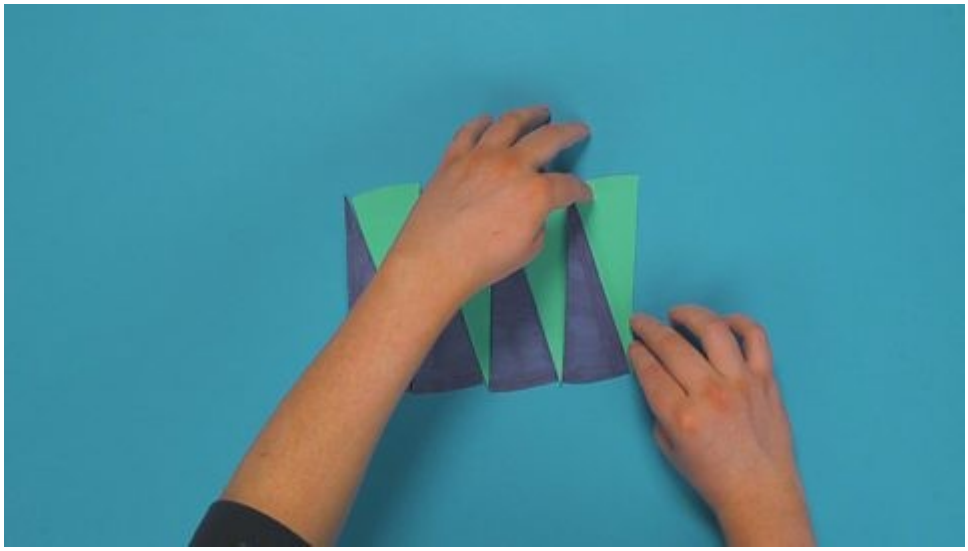


...the bottom edge of the rectangle is 1/2 of the circumference of the base of the cone. The height of the rectangle is the slant length...

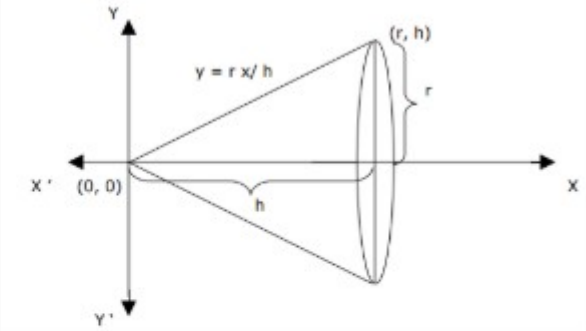








PROOF



Let us consider a right circular cone of radius r and height h.
The equation of the slant height is $y = \frac{r}{h}x$.

Then the volume of the cone is

$$\begin{aligned} S &= \int_0^h \pi y^2 dx \\ &= \pi \int_0^h \left(\frac{r}{h}x\right)^2 dx \\ &= \frac{\pi r^2}{h^2} \int_0^h x^2 dx \\ &= \frac{\pi r^2}{h^2} \cdot \frac{x^3}{3}\Big|_0^h \\ &= \frac{\pi r^2}{h^2} \cdot \frac{h^3}{3} \\ &= \frac{1}{3}\pi r^2 h. \square \end{aligned}$$

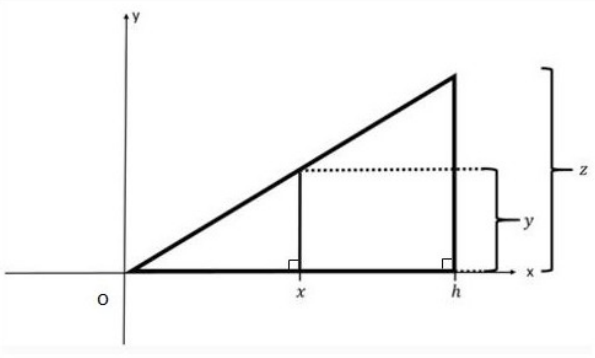
Pyramid Properties

Pyramid Volume

Derive the formula for the volume of a pyramid using calculus.

First, we want to find $A(x)$, where $A(x)$ is the function of the areas of the cross section of the pyramid.

Consider this:



Let h be the height of the solid, and z a constant such that $z^2 \propto A$. Also, let x and y be variables which would be used later to define $A(x)$.

Since $z^2 \propto A$, we can set $A = kz^2$, where (k) is another constant.

From the image above, it can be seen that both triangles are similar. So, finding the equation of y with respect to variable x, we have

$$\frac{z}{h} = \frac{y}{x} \Rightarrow y = \frac{xz}{h}.$$

Therefore,

$$A(x) = ky^2 = \frac{kz^2x^2}{h^2}.$$

Now, here comes the **integrating** part.

The volume of the object is

$$\begin{aligned}\int_b^a A(x)dx &= \int_0^h \frac{kz^2x^2dx}{h^2} \\ &= \frac{1}{3}kz^2h.\end{aligned}$$

Recall that $A = kz^2$, which gives

$$\frac{1}{3}kz^2h = \boxed{\frac{1}{3}Ah}.$$

Since the volume is based on the area of the cross section, the point at the top of the pyramid can be anywhere and this formula would still work. □

Sphere

Properties

Sphere Volume

At any given x, the incremental volume (δV) equals the product of the cross-sectional area of the disk at x and its thickness (δx):

$$\delta V \approx \pi y^2 \cdot \delta x.$$

The total volume is the summation of all incremental volumes:

$$V \approx \sum \pi y^2 \cdot \delta x.$$

In the limit as δx approaches zero,[8] this equation becomes:

$$V = \int_{-r}^{r} \pi y^2 \, dx.$$

At any given x, a right-angled triangle connects x, y and r to the origin; hence, applying the Pythagorean theorem yields:

$$y^2 = r^2 - x^2.$$

Using this substitution gives

$$V = \int_{-r}^{r} \pi \left(r^2 - x^2\right) dx,$$

which can be evaluated to give the result

$$V = \pi \left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r} = \pi \left(r^3 - \frac{r^3}{3}\right) - \pi \left(-r^3 + \frac{r^3}{3}\right) = \frac{4}{3}\pi r^3.$$

At any given radius r,[note 1] the incremental volume (δV) equals the product of the surface area at radius r ($A(r)$) and the thickness of a shell (δr):

$$\delta V \approx A(r) \cdot \delta r.$$

The total volume is the summation of all shell volumes:

$$V \approx \sum A(r) \cdot \delta r.$$

In the limit as δr approaches zero[8] this equation becomes:

$$V = \int_0^r A(r)\, dr.$$

Substitute V:

$$\frac{4}{3}\pi r^3 = \int_0^r A(r)\, dr.$$

Differentiating both sides of this equation with respect to r yields A as a function of r:

$$4\pi r^2 = A(r).$$

This is generally abbreviated as:

$$A = 4\pi r^2,$$

where r is now considered to be the fixed radius of the sphere.

References

Circle Properties

https://proofwiki.org/wiki/Perimeter_of_Circle

https://en.wikipedia.org/wiki/Area_of_a_circle#Modern_proofs

Cone Properties

https://www.bbc.co.uk/bitesize/articles/zqg7b82

https://brilliant.org/wiki/volume-cone/

Pyramid Properties

https://brilliant.org/wiki/volume-pyramid/

Sphere Properties

https://en.wikipedia.org/wiki/Sphere#Enclosed_volume

https://en.wikipedia.org/wiki/Sphere#Surface_area

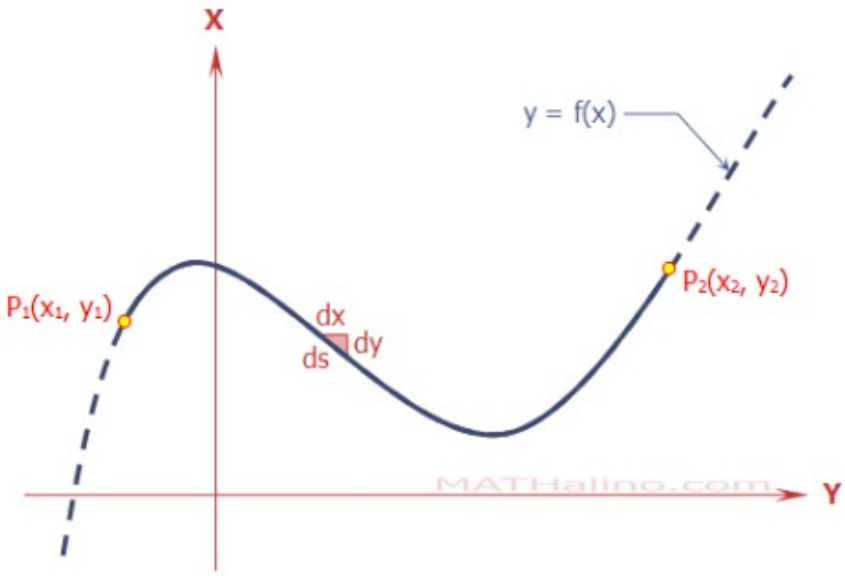
Review

Arc length formula

https://mathalino.com/reviewer/integral-calculus/length-arc-xy-plane-applications-integration

Apply Pythagorean theorem to the triangular strip shown in the figure:

$(ds)^2 = (dx)^2 + (dy)^2$ ← Equation (1)



Divide both sides of Equation (1) by $(dx)^2$

$$\frac{(ds)^2}{(dx)^2} = \frac{(dx)^2}{(dx)^2} + \frac{(dy)^2}{(dx)^2}$$

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

https://mathalino.com/reviewer/derivation-formulas/derivation-formula-total-surface-area-sphere-integration

#todo Paste

Volume in spherical coordinates

https://en.wikipedia.org/wiki/Spherical_coordinate_system

In terms of Jacobian

namely

$$\mathrm{d}V = \left|\frac{\partial(x, y, z)}{\partial(r, \theta, \varphi)}\right| \mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi = r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi = r^2\,\mathrm{d}r\,\mathrm{d}\Omega\,.$$

Where the solid angle is:

Thus the differential solid angle is

$$\mathrm{d}\Omega = \frac{\mathrm{d}S_r}{r^2} = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi.$$

Sphere Volume

Wikipedia Article on Sphere Volume (Calculus Proof)

An alternative formula is found using spherical coordinates, with volume element

$$dV = r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

so

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} r'^2 \sin\theta\, dr'\, d\theta\, d\varphi = 2\pi\int_0^{\pi}\int_0^{r} r'^2 \sin\theta\, dr'\, d\theta = 4\pi\int_0^{r} r'^2\, dr' = \frac{4}{3}\pi r^3.$$